equations. The computation was performed for a source flow mixing layer type DF chemical laser cavity using a stable implicit numerical procedure.[4] Seven vibrational levels of DF, each having 14 equilibrium rotational states, were considered. The chemical reactions were taken from Ref. 5, which are predominantly bimolecular. Three cases have been computed for illustration. The reference case corresponds to a cavity inlet pressure of $p_0 = 4.66$ Torr and an element length (centerline to centerline distance between a pair of fuel and oxidizer nozzles) of $L = 0.198$ cm. The second case is for $2p_0$ and $0.5L$, and the third for $0.5p_0$ and $2L$.

The results show that the specific power σ of the second and third cases $=0.991$ and 1.008, respectively, of the reference case. Hence specific power is essentially unaltered by the scaling. The small differences are due to pressure broadening effects, which are indeed minor.

The power flux distributions, normalized by the maximum value reached in the reference case, are shown in Fig. 1. It is seen that the power flux scales inversely with L, while the power cutoff distance scales directly with L.

The present analysis indicates that, even though laser flowfields are quite complicated and involve various nonlinear physical processes, unique and precise pressure-geometry scaling results can be reached. The restrictions, which have been delineated at the relevant points of the analysis, are surprisingly few and are easily satisfied under common operating conditions. Specifically, for combustor temperatures above ~ 1500 K, the effect of pressure variation on incomplete combustion is small. For cavity pressures under ~ 20 Torr, three-body reactions and pressure-broadening effects on gain are negligible.

Conclusions

In conclusion, the results of the present scaling analysis indicate that:

1) The conservation equations and constitutive relations for chemical laser flows are satisfied by the pressure-length scaling.

2) Flow similarity is achieved due to invariance in Reynolds number. Chemical similarity is obtained for bimolecular reactions. Radiative similarity is obtained as gain and intensity both vary with density.

3) Tradeoff between power flux and lasing mode length can be accomplished at a given specific power level by applying pressure-geometry scaling.

References

[1]Bird, R. B., Stewart, W. E., and Lightfoot, E. N., *Transport Phenomena*, John Wiley and Sons, New York, 1960.

[2]Gosman, A. D., Pun, W. M., Runchal, A. K., Spalding, D. B., and Wolfshtein, M., *Heat and Mass Transfer in Recirculating Flows*, Academic Press, New York, 1969.

[3]Thoenes, J., Hendricks, W. L., Kurzius, S. C., and Wang, F. C., "Advanced Laser Flow Analysis (ALFA) Theory and User's Guide," Lockheed Missiles and Space Co., Rept. AFWL-TR-78-19, Feb. 1979.

[4]Quan, V., Persselin, S. F., and Yang, T. T., "Computation of Reacting Flowfield with Radiation Interaction in Chemical Lasers," *AIAA Journal*, Vol. 21, Sept. 1983, pp. 1283-1288.

[5]Cohen, N. and Bott, J. F., "Review of Rate Data for Reactions of Interest in HF and DF Lasers," Aerospace Corp. Rept., TR-0083(3603)-2, 1982.

Numerical Evaluation of Line Integrals with a Logarithmic Singularity

John T. Katsikadelis* and Anthony E. Armenàkas†
Polytechnic Institute of New York
Brooklyn, New York

Introduction

IN solving two-dimensional boundary value problems by the boundary element method, line integrals of the following form are encountered[1-3]

$$\int_C \varphi(s) K(r) \mathrm{d}s_q \tag{1}$$

where C is the boundary element, i.e., a finite plane arc; $r = r(p,q) = |p-q|$ is the distance between any two points p and $q \in C$; the index q in the arc length element $\mathrm{d}s_q$ indicates that point q varies during integration while the point p remains constant; $\varphi(s)$ is a continuous function of the arc length $s = s(p,q)$ measured from point p; and the kernel function $K(r)$ has a logarithmic singularity. That is,

$$K(r) = \psi_1(r) + \psi_2(r) \ln r \tag{2}$$

The functions $\psi_1(r)$ and $\psi_2(r)$ are continuous. In mechanics, integral (1) represents a single-layer potential due to a material curve C of line density $\varphi(s)$ when the field point p is on the source.

For computer implementation of the boundary element method, the boundary element is approximated by a straight line or a parabolic arc.[3] In the first case, $r = s$ and integral (1) may be evaluated by either analytical integration, when the functions $\varphi(s)$, $\psi_1(r)$, and $\psi_2(r)$ are simple, or numerically using a special logarithmically weighted integration formula[3,4] when these functions are not simple. However, when the element is approximated by a higher-order curve, the evaluation of line integral (1) is more difficult. In this case, the integral has been evaluated by the method of subtracting the singularity.[5] In this method, a function is subtracted from the integrand that has a singularity of the same kind and can be integrated formally. This method has been employed in Ref. 6 for the numerical evaluation of integral (1) using the actual geometry of the element.

In this Note, a simple method is presented for the evaluation of line integral (1) by removing the singularity by means of integrating by parts when the curved element is approximated by a parabolic arc. This procedure can be easily programmed on a computer.

Received May 10, 1984; revision received Sept. 16, 1984.

*Associate Professor, Institute of Structural Analysis, National Technical University of Athens, Greece.

†Professor of Aerospace Engineering; also, Professor and Director of the Institute of Structural Analysis, National Technical University of Athens, Greece. Associate Fellow AIAA.

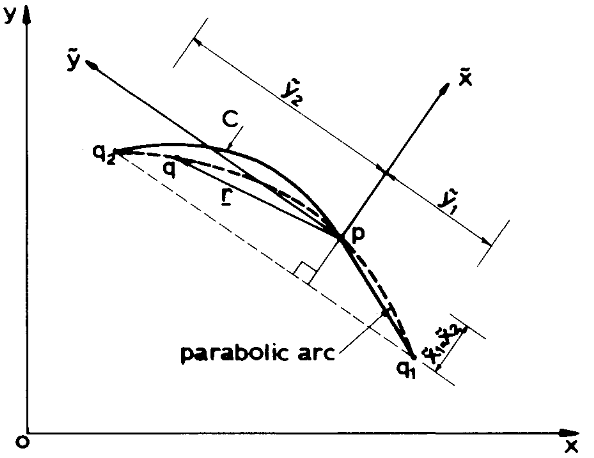

Fig. 1 Boundary element C.

Numerical Procedure

Consider a boundary element C extending from point q_1 to point q_2 with a fixed point p. The coordinates of points q_1, q_2, and p are given with respect to the system of axes Oxy (see Fig. 1). Moreover, consider a second system of axes $p\tilde{x}\tilde{y}$ with p as its origin, having the $\tilde{y}$ axis parallel to the line connecting the points q_1 and q_2.[7] The coordinates of points q_1, q_2, and p with respect to this system of axes can be established easily from their given coordinates with respect to the system of axes Oxy. The boundary element is approximated by a parabolic arc passing through the points q_1, q_2, and p. The equation of the parabolic arc referred to the system of axes $p\tilde{x}\tilde{y}$ has the form

$$\tilde{x}=\alpha_1\tilde{y}^2+\alpha_2\tilde{y} \tag{3}$$

The constants α_1 and α_2 in Eq. (3) are obtained from the known coordinates of points q_1 and q_2 (see Fig. 1) as

$$\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}=\begin{bmatrix}\tilde{y}_1^2 & \tilde{y}_1\\ \tilde{y}_2^2 & \tilde{y}_2\end{bmatrix}^{-1}\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\end{bmatrix}=\frac{1}{\tilde{y}_1\tilde{y}_2(\tilde{y}_1-\tilde{y}_2)}\begin{bmatrix}\tilde{x}_1\tilde{y}_2-\tilde{x}_2\tilde{y}_1\\ -\tilde{x}_1\tilde{y}_2^2+\tilde{x}_2\tilde{y}_1^2\end{bmatrix} \tag{4}$$

Integral (1) can be written as

$$\int_C \varphi(s)K(r)\mathrm{d}s=\int_C[\varphi(s)K(r)-\varphi(0)\psi_2(0)\ln r]\mathrm{d}s+\varphi(0)\psi_2(0)\int_C \ln r\mathrm{d}s \tag{5}$$

Notice that

$$\lim_{r\to 0}[\varphi(s)K(r)-\varphi(0)\psi_2(0)\ln r]=\lim_{r\to 0}\{\varphi(s)\psi_1(r)+[\varphi(s)\psi_2(r)-\varphi(0)\psi_2(0)]\ln r\}=\varphi(0)\psi_1(0)=\text{finite} \tag{6}$$

Consequently, the first integral on the right-hand side of Eq. (5) is not singular and, thus, it can be evaluated using any of the known numerical techniques. The required function $s(\tilde{y})$ is obtained from Eq. (10). To evaluate the second integral on the right-hand side of Eq. (5), we integrate it by parts and transform it to the $\tilde{x}$, $\tilde{y}$ coordinates. That is,

$$\int_C \ln r\mathrm{d}s=[g(\tilde{y})]_{\tilde{y}_1}^{\tilde{y}_2}-\int_{\tilde{y}_1}^{\tilde{y}_2}f(\tilde{y})\mathrm{d}\tilde{y} \tag{7}$$

Table 1 Percent error in the values of integral $\int_C \ln r\mathrm{d}s$ vs angle θ of the circular element

θ, deg	Exact value	Linear approximation	Parabolic approximation
30	−0.864379	-68.11×10^{-2}	0.54×10^{-2}
25	−0.799360	50.45×10^{-2}	0.28×10^{-2}
20	−0.717047	34.28×10^{-2}	0.13×10^{-2}
15	−0.612906	20.41×10^{-2}	0.03×10^{-2}
10	−0.479279	9.62×10^{-2}	0.02×10^{-2}
5	−0.300100	2.57×10^{-2}	~0
2	−0.152022	0.46×10^{-2}	~0
1	−0.088108	0.11×10^{-2}	~0

where

$$f(\tilde{y})=s(\tilde{y})\frac{\tilde{x}z+\tilde{y}}{r^2} \tag{8a}$$

$$g(\tilde{y})=s(\tilde{y})\ln r \tag{8b}$$

$$z=\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tilde{y}}=2\alpha_1\tilde{y}+\alpha_2 \tag{9a}$$

$$r=\sqrt{\tilde{x}^2+\tilde{y}^2} \tag{9b}$$

and

$$s(\tilde{y})=\int_0^{\tilde{y}}\sqrt{1+(2\alpha_1\tilde{y}'+\alpha_2)^2}\mathrm{d}y'=\frac{1}{4\alpha_1}[z\sqrt{1+z^2}+\ln(z+\sqrt{1+z^2})-\alpha_2\sqrt{1+\alpha_2^2}-\ln(\alpha_2+\sqrt{1+\alpha_2^2})] \tag{10}$$

Using L'Hospital's rule, it can be shown that

$$\lim_{\tilde{y}\to 0}f(\tilde{y})=f(0)=\sqrt{1+\alpha_2^2} \tag{11a}$$

$$\lim_{\tilde{y}\to 0}g(\tilde{y})=g(0)=0 \tag{11b}$$

Thus, the functions $f(\tilde{y})$ and $g(\tilde{y})$ are not singular and, consequently, the integral in Eq. (7) can be evaluated using any of the known numerical techniques.

When the boundary element is a straight line, the approximating curve is also a straight line. Thus, in this case, Eq. (4) yields α_1 and $\alpha_2=0$. Consequently,

$$\tilde{x}=0,\quad z=0,\quad s(\tilde{y})=\tilde{y},\quad r=|\tilde{y}|$$

Thus, the functions $f(\tilde{y})$ and $g(\tilde{y})$ are equal to

$$f(\tilde{y})=1 \tag{12a}$$

$$g(\tilde{y})=\tilde{y}\ln|\tilde{y}| \tag{12b}$$

Numerical Results and Conclusions

A subroutine program has been written using the aforedescribed numerical procedure for the evaluation of line integrals with a logarithmic singularity when the boundary element is approximated by a parabolic arc. In this subroutine, the regular integrals in Eqs. (5) and (7) are evaluated by the Gauss integration method. Numerical results are obtained using the aforementioned subroutine for the following integral:

$$I_1=\int_C \ln r\mathrm{d}s \tag{13}$$

where C is an arc of a circle.

In Table 1, the percent error in the values of Eq. (13) is presented for various values of the angle θ when it is evaluated numerically using both linear and parabolic approximations of the boundary element. C is the arc of a circle of unit radius and angle θ. In this case, Eq. (13) reduces to the Clausen's integral, that is

$$I_l = \int_C \ell n r \mathrm{d}s = \int_0^\theta \ell n \left(2\sin\frac{t}{2}\right) \mathrm{d}t$$

The values of this integral are tabulated in Ref. 8. Moreover, in this case, for linear approximation of the boundary element, the line integral [Eq. (13)] is evaluated analytically as

$$I_l = \int_C \ell n r \mathrm{d}s \simeq 2\sin\frac{\theta}{2}\left[\ell n\left(2\sin\frac{\theta}{2}\right) - 1\right]$$

From the numerical results, it is apparent that the accuracy in the values of the singular line integral (1) improves appreciably when the curved boundary element is approximated by a parabolic arc rather than when it is approximated by a straight line.

References

[1]Katsikadelis, J. T., "The Analysis of Plates on Elastic Foundation by the Boundary Integral Equation Method," Ph.D. Dissertation, Polytechnic Institute of New York, Brooklyn, New York, June 1982, pp. 81-87.

[2]Katsikadelis, J. T. and Armenàkas, A. E., "Plates on Elastic Foundation by the BIE Method," *ASCE Journal of Engineering Mechanics*, Vol. 110, No. 7, 1984, pp. 1086-1105.

[3]Brebbia, C. A., *The Boundary Element Method for Engineers*, 2nd ed., Pentech Press, London, Plymouth, 1980, pp. 52-58 and 187-188.

[4]Stroud, A. H. and Secrest, D. H., "Gaussian Quadrature Formulas," Prentice-Hall, Englewood Cliffs, N.J., 1966.

[5]Hartee, D. R., *Numerical Analysis*, 2nd ed., Clarendon Press, Oxford, England, 1958, p. 110.

[6]Christiansen, S., "Numerical Solution of an Integral Equation with a Logarithmic Kernel," *BIT*, Vol. 11, 1971, pp. 276-287.

[7]Katsikadelis, J. T. and Armenàkas, A. E., "Numerical Evaluation of Double Integrals with a Logarithmic or a Cauchy-Type Singularity," *Journal of Applied Mechanics*, Vol. 50, No. 3, 1983, pp. 682-684.

[8]Abramowitz, M. and Stegun, I. (eds.), *Handbook of Mathematical Functions*, 10th ed., Dover Publications, New York, 1972, pp. 1005-1006.